

# DIMENSIONS OF QUANTIZED TILTING MODULES

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ABSTRACT. Let  $U$  be the quantum group with divided powers at  $p$ -th root of unity for prime  $p$ . For any two-sided cell  $A$  in the corresponding affine Weyl group one associates tensor ideal in the category of tilting modules over  $U$ . In this note we show that for any cell  $A$  there exists tilting module  $T$  from the corresponding tensor ideal such that biggest power of  $p$  which divides  $\dim T$  is  $p^{a(A)}$  where  $a(A)$  is Lusztig's  $a$ -function. This result is motivated by the Conjecture of J. Humphreys in [3].

## 1. INTRODUCTION

Let  $G$  be a simply connected algebraic group. In [6] G. Lusztig proved existence of bijection between two finite sets: the set of two-sided cells in the affine Weyl group attached to  $G$  (this set is defined combinatorially) and the set of unipotent  $G$ -orbits. The proof in [6] is quite involved and this bijection remains rather mysterious. In [3] J. Humphreys suggested a natural conjectural construction for Lusztig's bijection using cohomology of tilting modules over algebraic groups in characteristic  $p > 0$  or, similarly, over quantum groups at a root of unity. In [10, 11] the author proved some partial results towards this Humphreys' Conjecture. Now this Conjecture is known to be true in a quantum group case thanks to the (unpublished) work of R. Bezrukavnikov.

For an element  $w$  of a finite Weyl group  $W_f$  one defines number  $a(w)$  as Gelfand-Kirillov dimension of highest weight simple module  $L(w \cdot 0)$  over the corresponding semisimple Lie algebra. Generalizing this G. Lusztig defined  $a$ -function on any Coxeter group, see [4]. The  $a$ -function takes constant value on any two-sided cell and appears to be very useful for the theory of cells in Coxeter groups. J. Humphreys suggested that his construction of Lusztig's bijection is compatible with the theory of  $a$ -function in the following way: dimension of any tilting module corresponding to two-sided cell  $A$  is divisible by  $p^{a(A)}$  and generically is not divisible by higher power of  $p$  (here  $p$  is characteristic of field in a case of algebraic group and order of root of unity in a case of quantum group). In [11] the author proved that the first statement (divisibility of dimensions by  $p^{a(A)}$ ) is a consequence of Humphreys' Conjecture, so it is a consequence of Bezrukavnikov's work. The second statement (generic indivisibility by  $p^{a(A)+1}$ ) seems to be harder. The main result of this note is that in quantum group case for any cell  $A$  there exists tilting module  $T$  corresponding to  $A$  such that its dimension is not divisible by  $p^{a(A)+1}$ . So we determine  $p$ -component of dimension of certain tilting modules what seems to be of some interest independently of Humphreys' Conjecture.

We will follow the notations of [9]. Let  $(Y, X, \dots)$  be a simply connected root datum of finite type. Let  $p$  be a prime number bigger than the Coxeter number  $h$ .

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Let  $\zeta$  be a primitive  $p$ -th root of unity in  $\mathbb{C}$ . Let  $U$  be the quantum group with divided powers associated to these data. Let  $\mathcal{T}$  be the category of *tilting* modules over  $U$ , see e.g. [1]. Recall that any tilting module is a sum of indecomposable ones, and indecomposable tilting modules are classified by their highest weights, see *loc. cit.* Let  $X_+$  be the set of dominant weights, and for any  $\lambda \in X_+$  let  $T(\lambda)$  denote the indecomposable tilting module with highest weight  $\lambda$ . The tensor product of tilting modules is again a tilting module.

Let us introduce the following preorder relation  $\leq_T$  on  $X_+$ :  $\lambda \leq_T \mu$  iff  $T(\lambda)$  is a direct summand of  $T(\mu) \otimes$  (some tilting module). We say that  $\lambda \sim_T \mu$  if  $\lambda \leq_T \mu$  and  $\mu \leq_T \lambda$ . Obviously,  $\sim_T$  is an equivalence relation on  $X_+$ . The equivalence classes are called *weight cells*. The set of weight cells has a natural order induced by  $\leq_T$ . It was shown in [10] that the partially ordered set of weight cells is isomorphic to the partially ordered set of *two-sided cells* in the affine Weyl group  $W$  associated with  $(Y, X, \dots)$  ( $W$  is a semidirect product of the finite Weyl group  $W_f$  with the dilated coroot lattice  $pY$ ).

Let  $G$  and  $\mathfrak{g}$  be the simply connected algebraic group and the Lie algebra (both over  $\mathbb{C}$ ) associated to  $(Y, X, \dots)$ , and let  $\mathcal{N}$  be the *nilpotent cone* in  $\mathfrak{g}$ , i.e. the variety of *ad*-nilpotent elements. It is well known that  $\mathcal{N}$  is a union of finitely many  $G$ -orbits called *nilpotent orbits*. Using the theory of *support varieties* one defines *Humphreys map*  $\mathcal{H} : \{\text{the set of weight cells}\} \rightarrow \{\text{the set of closed } G\text{-invariant subsets of } \mathcal{N}\}$ , see [11]. The construction is as follows: it is known that cohomology ring  $H^\bullet(u)$  of small quantum group  $u \subset U$  is isomorphic to the ring of regular functions on  $\mathcal{N}$  (this is a Theorem due to V. Ginzburg and S. Kumar, see [2]); now let  $A$  be a weight cell and take any weight  $\lambda \in A$ , then  $\text{Ext}^\bullet(T(\lambda), T(\lambda))$  is naturally a module over  $H^\bullet(u)$ , so it can be considered as a coherent sheaf on  $\mathcal{N}$ , finally Humphreys map  $\mathcal{H}(A)$  is just support of this sheaf. The Conjecture due to J. Humphreys says that the image of map  $\mathcal{H}$  consists of irreducible varieties, i.e. the closures of nilpotent orbits; moreover J. Humphreys conjectured that this map coincides with *Lusztig's bijection* between the set of two-sided cells in the affine Weyl group and the set of nilpotent orbits, see [3] and [11]. In particular, Humphreys map should preserve *Lusztig's a-function*; this function is equal to half of codimension in  $\mathcal{N}$  of the nilpotent orbit and is defined purely combinatorially on the set of two-sided cells, see [6]. The aim of this note is to show that the Humphreys map does not decrease the *a-function*: for a weight cell  $A$  corresponding to a two-sided cell  $\underline{A}$  in  $W$  we have the inequality  $\text{codim}_{\mathcal{N}} \mathcal{H}(A) \geq a(\underline{A})$ . This inequality follows easily from the definition of  $\mathcal{H}$ , Theorem 4.1 in [11] and the Main Theorem below:

**Main Theorem.** *Let  $A$  be a weight cell corresponding to a two-sided cell  $\underline{A}$  in the affine Weyl group. Then there exists a weight cell  $B \leq_T A$  and a regular weight  $\lambda \in B$  such that  $\dim T(\lambda)$  is not divisible by  $p^{a(\underline{A})+1}$  provided  $p$  is sufficiently large.*

**Remark.** It follows from the Humphreys' Conjecture proved by Bezrukavnikov that in the Main Theorem  $B = A$ .

The proof of this Theorem is based on formulas for characters of indecomposable tilting modules obtained by W. Soergel in [12, 13]. In what follows we will freely use notations and results from [12] and [11].

**Warning.** The character formulas for tilting modules use certain Kazhdan-Lusztig elements in the Iwahori-Hecke algebra of  $W$ , and in modules thereof. The Iwahori-Hecke algebra is an algebra over  $\mathbb{Z}[v, v^{-1}]$ , and we will only need its specialization at  $v = 1$ . So all the notions related with it (e.g. Kazhdan-Lusztig bases) will be understood in the specialization  $v = 1$ .

## 2. PROOF OF THE MAIN THEOREM.

Let  $\leq$  denote the Bruhat order on the affine Weyl group  $W$ . For any  $x \in W$  let  $(-1)^x$  denote sign of  $x$ , that is  $(-1)^{l(x)}$  where  $l(x)$  is length of  $x$ .

2.1. We may and will suppose that our root system  $R$  is irreducible. Let  $S$  be the set of simple reflections in the affine Weyl group  $W$ . For any  $s \in S$  let  $W_s$  be the parabolic subgroup generated by  $S - \{s\}$ . The subgroup  $W_s$  is finite. There exists a unique point  $p\mu_s \in X \otimes_{\mathbb{Z}} \mathbb{Q}$  invariant under the  $W_s$ -action. In general,  $\mu_s \notin X$ , but the denominators of its coordinates contain only bad primes for  $R$ .

In particular, let  $s_a \in S$  be the unique affine reflection. Then  $W_{s_a} = W_f$  is the finite Weyl group. There exists a natural projection  $W \rightarrow W_f, x \mapsto \bar{x}$ . This projection embeds all the subgroups  $W_s$  into  $W_f$ .

Recall from [12] that the set  $W^f$  of minimal length representatives of cosets  $W_f \backslash W$  is identified with the set of dominant alcoves.

2.2. Recall that any two-sided cell of  $W$  intersects nontrivially some  $W_s$ , see [6]. In the group algebra of  $W_s$  there are two remarkable bases: the Kazhdan-Lusztig base  $\tilde{H}_w, w \in W_s$ , and the dual Kazhdan-Lusztig base  $\underline{H}_w, w \in W_s$  (notations from [12]). Recall that  $\underline{H}_w = \sum_{x \leq w} p_{x,w} x$  and  $\tilde{H}_w = \sum_{x \leq w} p_{x,w} (-1)^{xw} x$  where  $p_{x,w}$  are the values at 1 of Kazhdan-Lusztig polynomials.

Let  $V = X \otimes_{\mathbb{Z}} \mathbb{R}$  be the reflection representation of  $W_f$ . For any  $s \in S$  the restriction of  $V$  to  $\overline{W}_s$  is isomorphic to the reflection representation  $V_s$  of  $W_s$ .

We refer the reader to [4] for definition and properties of Lusztig's  $a$ -function. This function is defined on the set of elements of a Coxeter group and takes values in  $\mathbb{N} \cup \infty$ . We will use following properties of  $a$ -function:

- (i)  $a$ -function is constant on any two-sided cell, see [4] 5.4.
- (ii) Suppose that  $w \in W_0 \subset W$  where  $W_0$  is a parabolic subgroup of  $W$ . Then values of  $a$ -function of  $w$  calculated with respect to Coxeter groups  $W_0$  and  $W$  coincide, see [5] 1.9 (d).
- (iii) Let  $w \in W_s$ . The element  $\tilde{H}_w$  acts trivially on  $S^i(V_s)$  for  $i < a(w)$ , see [7]. The space  $S^{a(w)}(V_s)$  contains exactly one irreducible component (special representation) such that elements  $\tilde{H}_{w'}, w' \sim_{LR} w$ , act nontrivially on it, see *loc. cit.* Moreover, these elements generate an action of the full matrix algebra on this component, see [8] Chapter 5. We will say that this special representation corresponds to  $w$ .

**Convention.** The equivalence relation  $\sim_{LR}$  depends on the ambient group, e.g. if  $w_1, w_2 \in W_s$  then  $w_1 \sim_{LR} w_2$  in  $W$  does not imply  $w_1 \sim_{LR} w_2$  in  $W_s$ . In what follows the equivalence relation  $\sim_{LR}$  is considered with respect to  $W_s$ . In spite of this we apply the notation  $\leq_{LR}$  with respect to  $W$ . We hope that this does not cause ambiguity in what follows.

2.3. Let  $\Delta(\lambda) = \prod_{\alpha \in R_+} \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$  be the Weyl polynomial. For any  $w \in W_s$  and  $y \in W_f$  let us consider the following polynomial in  $\lambda$  and  $\mu$ :

$$\Delta(y, W_s, w, \mu, \lambda) = \sum_{x \leq w} p_{x,w} \Delta(\mu + y\bar{x}y^{-1}\lambda).$$

**Lemma.** *The lowest degree term of  $\Delta(y, W_s, w, \mu, \lambda)$  in  $\mu$  has degree  $\geq a(w)$ .*

**Proof.** It is well known that the polynomial  $\Delta(\lambda)$  is skew-symmetric with respect to the  $W_f$ -action:  $\Delta(y\lambda) = (-1)^y \Delta(\lambda)$ . Hence

$$\Delta(y, W_s, w, \mu, \lambda) = \Delta(1, W_s, w, y^{-1}\mu, y^{-1}\lambda).$$

So for the proof of the Lemma it is enough to consider the case  $y = 1$ . Using skew-symmetry of  $\Delta(\lambda)$  with respect to the  $W_f$ -action once again, we have:

$$(*) \quad \Delta(1, W_s, w, \mu, \lambda) = \sum_{x \leq w} p_{x,w} (-1)^x \Delta(\bar{x}^{-1}\mu + \lambda).$$

The element  $\sum_{x \leq w} p_{x,w} (-1)^x x^{-1}$  equals  $\tilde{H}_{w^{-1}}$ , see e.g. [12], proof of the Theorem 2.7. Since  $a(w) = a(w^{-1})$  (see [4]), the result follows from 2.2.

2.4. Now we consider  $\Delta(\mu + \lambda)$  as polynomial in two variables  $\mu, \lambda \in V$ . The action of the Weyl group  $W_f$  on the space  $S^\bullet(V \oplus V)$  of all polynomials in two variables  $\mu$  and  $\lambda$  via the variable  $\mu$  is well-defined and preserves degrees of polynomials with respect to both  $\mu$  and  $\lambda$ .

**Lemma.** *Let  $W_s$  act on the polynomial  $\Delta(\mu + \lambda)$  by the rule  $x\Delta(\mu + \lambda) = \Delta(\bar{x}\mu + \lambda)$ . Then the representation generated by the summand of degree  $a(w)$  in  $\mu$  contains the special representation corresponding to  $w^{-1}$ .*

**Proof.** Let  $E_1 \subset S^{a(w)}(V)$  be the special representation of  $W_s$  corresponding to  $w^{-1}$ . According to [7] sect.3, the  $W_f$ -representation  $E$  generated by  $E_1$  is irreducible, occurs with multiplicity 1 in the space of polynomials of degree  $a(E_1) = a(w)$  and does not occur in the spaces of polynomials of lower degree. Moreover,  $E$  lies in the space of harmonic polynomials which is identified with the cohomology of the flag variety  $H^{2a(w)}(G/B)$ . Hence the Lemma is reduced to the following statement:

2.4.1. **Lemma.** (R. Bezrukavnikov) *Let  $W_f$  act on the polynomial  $\Delta(\mu + \lambda)$  by the rule  $x\Delta(\mu + \lambda) = \Delta(x\mu + \lambda)$ . Then the representation generated by the summand of degree  $i$  in  $\mu$  contains any irreducible constituent of  $H^{2i}(G/B)$ .*

**Proof.** We identify cohomology space  $H^\bullet(G/B \times G/B)$  with the space of harmonic polynomials in two variables  $\mu$  and  $\lambda$  (with respect to the group  $W \times W$ ). It is well known that diagonal class is represented by  $\Delta(\mu + \lambda)$ . Using Poincaré duality we identify  $H^\bullet(G/B \times G/B)$  with  $\text{End}(H^\bullet(G/B))$  (this identification is not  $W \times W$ -equivariant since fundamental class is  $W$ -antiinvariant but it is  $W$ -equivariant with respect to the action of first copy of  $W$ ). Now any vector  $v \in H^\bullet(G/B)$  defines  $W$ -equivariant map  $\text{End}(H^\bullet(G/B)) \rightarrow H^\bullet(G/B)$ ,  $x \mapsto xv$  where  $W$  acts on  $\text{End}(H^\bullet(G/B)) = H^\bullet(G/B) \otimes (H^\bullet(G/B))^*$  via the first factor and under this map  $1 \mapsto v$ . The diagonal class  $\Delta(\mu + \lambda)$  corresponds to  $1 \in \text{End}(H^\bullet(G/B))$  and the summand of degree  $i$  in  $\mu$  corresponds to  $1 \in \text{End}(H^{2i}(G/B))$ . The result follows.

2.5. Let  $N$  denote the degree of the polynomial  $\Delta(\lambda)$ .

**Lemma.** *Let us fix  $\mu$  such that  $\langle \mu, \alpha^\vee \rangle \neq 0$  for any  $\alpha \in R$ . Then there exists  $w' \in W_s$ ,  $w' \sim_{LR} w$ , such that the summand of  $\Delta(y, W_s, w', \mu, \lambda)$  of degree  $N - a(w)$  in  $\lambda$  is nontrivial.*

**Proof.** We may and will assume that  $y = 1$ . By the formulae (\*) we have  $\Delta(1, W_s, w_1, \mu, \lambda) = \tilde{H}_{w_1^{-1}} \Delta(\mu + \lambda)$  where  $W_s$  acts on  $\Delta(\mu + \lambda)$  via the variable  $\mu$ . Since elements  $\tilde{H}_{w_1^{-1}}$ ,  $w_1 \sim_{LR} w$  generate an action of the full matrix algebra on the special representation corresponding to  $w^{-1}$  by 2.2 the Lemma 2.4 show

that the set of summands of degree  $a(w)$  in  $\mu$  of  $\tilde{H}_{w_1^{-1}}\Delta(\mu + \lambda)$  where  $w_1$  runs through all  $w_1 \sim_{LR} w$  contains a basis over the field of rational functions in  $\lambda$  of special representation corresponding to  $w^{-1}$ . Evidently these summands are exactly summands of  $\Delta(1, W_s, w_1, \mu, \lambda)$  of degree  $N - a(w)$  in the variable  $\lambda$ .

Our Lemma claims that this set contains at least one nonzero element when we specialize  $\mu$  to any weight satisfying conditions of the Lemma. Consider ideal generated by this set in a ring of polynomials in  $\mu$  with coefficients which are rational functions in  $\lambda$ . Evidently, the Lemma is a consequence of the following statement:

**2.5.1. Lemma.** *Let  $U$  be an irreducible  $W_f$ -submodule of  $S^\bullet(V)$  not contained in  $(S^+(V))^{W_f}$ . In other words,  $U$  projects nontrivially to  $S^\bullet(V)/(S^+(V))^{W_f} = H^{2\bullet}(G/B)$ . Then the zero set of the ideal of  $S^\bullet(V)$  generated by  $U$  is contained in the union of hyperplanes  $\langle \mu, \alpha^\vee \rangle = 0, \alpha \in R$ .*

**Proof.** Evidently, the ideal generated by  $U$  is  $W_f$ -invariant. By Poincaré duality for any  $0 \neq v \in H^i(G/B)$  there exists  $v' \in H^{2N-i}(G/B)$  such that  $vv'$  represents fundamental class of  $G/B$ . Hence the ideal generated by  $U$  contains an element  $\omega \in S^N(V)$  which projects nontrivially on  $H^{2N}(G/B)$ . The alternation  $\omega' = \frac{1}{|W_f|} \sum_{w \in W_f} (-1)^w w(\omega)$  is also contained in our ideal and projects nontrivially on  $H^{2N}(G/B)$ . But  $\omega'$  should be a nonzero multiple of Weyl polynomial  $\Delta(\lambda)$  since Weyl polynomial is unique up to scalar  $W$ -antiinvariant in  $S^N(V)$ . The Lemma is proved.

**2.6.** Let  $\underline{A} \subset W$  be a two-sided cell. Choose  $W_s$  such that  $W_s \cap \underline{A} \neq \emptyset$  (this is possible by [6] Theorem 4.8(d)). Let us fix  $w_1 \in W_s \cap \underline{A}$ . We choose  $y \in W^f$  minimal with the property:

(\*\*) For some  $w \in W_s$  such that  $w \sim_{LR} w_1$  the summand of  $\Delta(\bar{y}, W_s, w, y\mu_s, \bar{y}\lambda)$  of degree  $N - a(w)$  in  $\lambda$  is nonzero.

By Lemma 2.5 such  $y$  exists since there exists  $y \in W^f$  such that  $y\mu_s$  lies strictly inside the dominant Weyl chamber.

In the following Lemma we use notations of [12].

**Lemma.** *Let  $y \in W^f$  and  $w \in W_s$  be as above. Then the element  $\underline{N}_y \underline{H}_w \in \mathcal{N}$  is a sum of elements  $\underline{N}_x, x \leq_{LR} \underline{A}$ , with positive integral coefficients, and hence can be considered as the character of tilting module in a regular block.*

**Proof.** By the formulae in the end of Proposition 3.4 of [12] we have:

$$\underline{N}_1 \underline{H}_x = \begin{cases} \underline{N}_x & \text{if } x \in W^f \\ 0 & \text{if } x \notin W^f. \end{cases}$$

So,  $\underline{N}_y \underline{H}_w = \underline{N}_1 \underline{H}_y \underline{H}_w$  and the Lemma follows from the definition of cells, together with the positivity properties of multiplication in the Iwahori-Hecke algebra, see e.g. [4] §3.

**2.7. Proof of the main Theorem.** We can rewrite the element  $\underline{N}_y \underline{H}_w$  as

$$\underline{N}_y \underline{H}_w = \underline{N}_1 \sum_{y_1 \in W^f, y_1 \leq y} n_{y_1, y} \sum_{x \leq w} p_{x, w} H_{y_1 x}.$$

Let  $\lambda_1$  be a regular weight from the fundamental alcove. The dimension of the tilting module  $T$  in the linkage class of  $\lambda_1$  with character given by  $\underline{N}_y \underline{H}_w$  is equal

to

$$\sum_{y_1 \in W^f, y_1 \leq y} n_{y_1, y} \sum_{x \leq w} p_{x, w} \Delta(y_1 x \cdot \lambda_1 + \rho).$$

Now let us write  $\lambda_1 = -\rho + p\mu_s + \lambda$ .

We have

$$\begin{aligned} \dim T &= \sum_{y_1 \leq y} n_{y_1, y} \sum_{x \leq w} p_{x, w} \Delta(y_1 p\mu_s + \overline{y_1} x \lambda) = \\ &= \sum_{y_1 \leq y} n_{y_1, y} \Delta(\overline{y_1}, W_s, w, y_1 p\mu_s, \overline{y_1} \lambda). \end{aligned}$$

According to (\*\*), for some  $w \sim_{LR} w_1$  the polynomial  $\dim T$  has nonvanishing summand of degree  $N - a(\underline{A})$  in  $\lambda$ . Hence, for  $p \gg 0$  it is possible to choose such  $\lambda$  that this summand is not divisible by  $p$  and  $\lambda_1$  lies in the lowest alcove. The Main Theorem is proved.

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